Backward Stochastic Differential Equations Associated with the Vorticity Equations

A. B. Cruzeiro and Z. M. Qian
Universidade de Lisboa and Oxford University

April 5, 2013

Abstract

In this paper, we derive a non-linear version of the Feynman-Kac formula for the solutions of the vorticity equation in dimension 2 with space periodic boundary conditions. We prove the existence (global in time) and uniqueness for a stochastic terminal value problem associated with the vorticity equation in dimension 2.

1 Introduction

The Feynman-Kac formula, in its original form derived from the idea of path integration in Feynman's PhD thesis (which is now available in a new print [7]), is a representation formula for solutions of Schöingder's equations, and in the hand of Kac, is an explicit formula written in terms of functional integrals with respect to the Wiener measure, the law of Brownian motion.

Bismut [3], Pardoux-Peng [11] and Peng [12], by utilizing Itô's lemma together with Itô's martingale representation, have obtained an interesting non-linear version of Feynman-Kac's formula for solutions of semi-linear parabolic equations in terms of backward stochastic differential equations (BSDE). The goal of the present paper is to derive a Feynman-Kac formula for solutions of the Navier-Stokes equations in the same spirit of Bismut and Pardoux-Peng [11], and to study the random terminal problem of the stochastic differential equations associated with the vorticity equations.

The main idea contained in [3], [11] may be described as the following. Let $u(t,x) = (u^1(t,x),\ \cdots,\ u^m(t,x))$ be a smooth solution to the Cauchy initial value problem of the following system of semi-linear parabolic equations

$$\frac{\partial}{\partial t}u^i - \nu\Delta u^i + f^i(u,\nabla u) = 0, \;\; u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^d \tag{1.1}$$

where $i = 1, \cdots, m$, and $\nu > 0$ a constant. Let $B = (B^1, \cdots, B^d)$ be the standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $x \in \mathbb{R}^d$ and $T > 0$. Let us read the solution

u along Brownian motion B. More explicitly, let $Y_t = u(T-t, \sqrt{2\nu}B_t + x)$ for $t \in [0,T]$ and $Z_t = \nabla u(T-t, \sqrt{2\nu}B_t + x)$, and apply Itô's formula to u and B to obtain

$$\begin{array}{c} Y_T - Y_t = \int_t^T f(Y_s, Z_s)ds + \sqrt{2\nu}\int_t^T Z_s \cdot dB_s, \\ Y_T = u_0(B_T). \end{array} \tag{1.2}$$

In literature, (1.1) may be written in differential form

$$dY = f(Y,Z)dt + \sqrt{2\nu}Z \cdot dB, \; Y_T = \xi, \tag{1.3}$$

where the arguments s, t etc. are suppressed if no confusion may arise. The differential equation above is an example of backward stochastic differential equations, where the terminal value $Y_T = \xi$ is given. The function f appearing on the right hand side of (1.3) is called the (non-linear) *driver*.

Pardoux-Peng [11] made an important observation. If the non-linear driver f in BSDE (1.3) is globally Lipschitz continuous, then there is a unique adapted solution pair (Y, Z) satisfying (1.3) for a random terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, which is not necessary in the form of $u_0(B_T)$. The solution u and its gradient ∇u in turn can be represented in terms of (Y, Z). This representation may be considered as a nonlinear extension of Feynman-Kac's formula to semi-linear parabolic equations.

More recently, Kobylanski [8], Delarue [6], Briand-Hu [4], Tevzadze [13], and etc. have extended Pardoux-Peng's result to some BSDEs with non-linear drivers of quadratic growth. These papers however mainly deal with scalar BSDEs only, which corresponds to semi-linear scalar parabolic equations. It remains largely an open problem whether the BSDE approach may be applied to non-parabolic type of partial differential equations. We study in the present paper a class of backward stochastic differential equations which arise from the vorticity formulation of the Navier-Stokes equations, hence provide Feynman-Kac type formula for solutions of the Navier-Stokes equations.

Relations between the Navier-Stokes equation and foward-backward stochastic differential equations formulated in the group of diffeomorphisms were introduced in [5].

2 The vorticity equation

Let us describe a class of (infinite dimensional) backward stochastic differential equations associated with the study of Navier-Stokes equations.

The 2D Navier-Stokes equations (without external force) are the partial differential equations which describe the motion of fluids

$$\begin{array}{c} \frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{array} \tag{2.1}$$

where $u = (u^1, u^2)$ is the velocity field, ν the viscosity constant and p the pressure. The mathematical study of the Navier-Stokes is interesting by its own, and even the simplest situation where the space periodic condition is supplied is of interest.

Suppose that $u(0,x) = \varphi(x)$ is a smooth vector field with period one, that is, $\varphi(x+e_i) = \varphi(x)$ for all $x \in \mathbb{R}^2$, where $e_1 = (1,0)$ and $e_2 = (0,1)$ the standard basis in $\mathbb{R}^2$. Then, the unique solution (u,p) to the 2D Navier-Stokes equation is smooth on $(0,\infty) \times \mathbb{R}^2$ and periodic in space variables, so that $u(t,x+e_i) = u(t,x)$ and $p(t,x+e_i) = p(t,x)$ for all $t > 0$, $x \in \mathbb{R}^2$, $i = 1,2$.

Let

$$\omega = \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2} \tag{2.2}$$

be the vorticity of u, which is a scalar function in dimension two, and thus the evolution equation for ω is a scalar partial differential equation

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + u \cdot \nabla \omega = 0. \tag{2.3}$$

Equation (2.3) is called the vorticity equation is equivalent to the Navier-Stokes equation. The relationship between the scalar function ω and the associated vector field u is determined by the Poisson equations

$$\Delta u^1 = -\frac{\partial \omega}{\partial x^2} \quad \text{and} \quad \Delta u^2 = \frac{\partial \omega}{\partial x^1}. \tag{2.4}$$

By (2.2) the average of $\omega(t,x)$

$$\int_{[0,1)^2} \omega(t,x)dx = 0 \quad \text{for all } t \geq 0, \tag{2.5}$$

so that (2.4) has a unique periodic (with period one) solution $u = (u^1, u^2)$. We define linear operators $K_i : \omega \to u^i$ (where $i = 1,2$) and $K : \omega \to u$ by solving the Poisson equations (2.4), where ω is a real function with period one and mean zero.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2D torus equipped with the standard metric and the Lebesgue measure. We may identify tensor fields in $\mathbb{R}^2$ with period one canonically with the corresponding tensor fields on $\mathbb{T}^2$. For example

$$L^2(\mathbb{T}^2) = \{f \in L^2_{\text{loc}}(\mathbb{R}^2) : f(\cdot + e_i) = f(\cdot) \text{ for } i = 1,2\} \cap L^2([0,1)^2).$$

If $f \in L^2(\mathbb{T}^2)$ then

$$f(x) = \sum_{k \in \mathbb{Z}^2} e^{2\pi\sqrt{-1}\langle k,x\rangle} \hat{f}(k) \tag{2.6}$$

where

$$\hat{f}(k) = \int_{[0,1)^2} e^{-2\pi\sqrt{-1}\langle k,y\rangle} f(y)dy, \quad k \in \mathbb{Z}^2 \tag{2.7}$$

is the Fourier transform of f, $\langle \cdot, \cdot \rangle$ denotes the scalar product in Euclidean spaces.

Lemma 2.1 *(Green's formula) Consider the Poisson equation*

$$\Delta g = -f \quad \textit{in } \mathbb{T}^2, \quad \int_{\mathbb{T}^2} g(y)dy = 0, \tag{2.8}$$

where $\int_{\mathbb{T}^2} f(y)dy = 0$ *and* $f \in L^2(\mathbb{T}^2)$. *Then the unique solution of the problem (2.8) is given by*

$$g(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{e^{2\pi\sqrt{-1}\langle k,x \rangle}}{4\pi|k|^2} \hat{f}(k) \ . \tag{2.9}$$

Our first goal is to derive a probabilistic representation for ω in terms of Brownian motion. To this end we set up the probability setting with which we are going to work with. Let $B = (B^1, B^2)$ be a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, and define

$$\begin{array}{rcl} Y(w,t,x) & = & \omega(T-t, x+\sqrt{2v}B_t(w)), \\ Z^1(w,t,x) & = & \frac{\partial \omega}{\partial x^1}(T-t, x+\sqrt{2v}B_t(w)), \\ Z^2(w,t,x) & = & \frac{\partial \omega}{\partial x^2}(T-t, x+\sqrt{2v}B_t(w)) \end{array}$$

for $(w,t,x) \in \Omega \times [0,\infty) \times \mathbb{R}^2$. We will often suppress the random element w from our notations, and write $Y(t,x)$, Y_t or simply by Y for $Y(w,t,x)$, if no confusion is possible.

Let $\psi = \frac{\partial \varphi_2}{\partial x^1} - \frac{\partial \varphi_1}{\partial x^2}$ be the vorticity of the initial velocity $\varphi \equiv u_0$, and $\xi(x) = \psi(x + \sqrt{2v}B_T)$. Then, it is clear that ξ is smooth and periodic in x (with again period one). According to Itô's formula

$$\begin{aligned} \xi(x) - Y(t,x) \;\; = \;\; & \sqrt{2v}\int_t^T \langle \nabla\omega(T-s, x+\sqrt{2v}B_s), dB_s \rangle \\ & + \int_t^T \left(-\frac{\partial \omega}{\partial s} + \nu\Delta\omega \right)(T-s, x+\sqrt{2v}B_s)ds. \end{aligned} \tag{2.10}$$

Now, by utilizing the vorticity equation (2.3): substitute $-\frac{\partial \omega}{\partial s} + \nu\Delta\omega$ by $u \cdot \nabla\omega$ to obtain

$$\xi(x) - Y(t,x) = \int_t^T \langle Z(s,x), X(s,x) \rangle ds + \sqrt{2v}\int_t^T \langle Z(s,x), dB_s \rangle \tag{2.11}$$

where for simplicity we have set

$$X(t,x) = u(T-t, x+\sqrt{2v}B_t)$$

which is continuous in t, smooth in x, and periodic in x. Next, we wish to rewrite $X(t,x)$ in terms of Y and Z. To this end, it is a good idea to introduce some notions in Fourier analysis, and establish several notations which will be used in what follows.

Let us apply Green's formula to the vorticity ω of u. According to (2.4) and (2.9), we have

$$u^1(x) = \frac{\sqrt{-1}}{2} \sum_{k=(k_1,k_2) \in \mathbb{Z}^2, k \neq 0} \frac{k_2}{|k|^2} e^{2\pi\sqrt{-1}\langle k,x \rangle} \widehat{\omega}(k) \tag{2.12}$$

and

$$u^2(x) = -\frac{\sqrt{-1}}{2} \sum_{k=(k_1,k_2) \in \mathbb{Z}^2, k \neq 0} \frac{k_1}{|k|^2} e^{2\pi\sqrt{-1}\langle k,x \rangle} \widehat{\omega}(k). \tag{2.13}$$

In other words

$$\widehat{u^1}(k) = \frac{\sqrt{-1}}{2}\frac{k_2}{|k|^2}\widehat{\omega}(k) \text{ and } \widehat{u^2}(k) = -\frac{\sqrt{-1}}{2}\frac{k_1}{|k|^2}\widehat{\omega}(k), \quad k \neq 0. \tag{2.14}$$

Hence

$$\begin{aligned} X^1(t,x) &= u^1(T-t, x+\sqrt{2v}B_t) \\ &= \frac{\sqrt{-1}}{2}\sum_{k=(k_1,k_2)\in\mathbb{Z}^2, k\neq 0}\frac{k_2}{|k|^2}e^{2\pi\sqrt{-1}\langle k, x+\sqrt{2v}B_t\rangle}\omega\widehat{(T-t},\cdot)(k). \end{aligned} \tag{2.15}$$

On the other hand

$$\begin{aligned} \widehat{Y(t,\cdot)}(k) &= \int_{[0,1)^2} e^{-2\pi\sqrt{-1}\langle k,y\rangle}\omega(T-t, y+\sqrt{2v}B_t)dy \\ &= e^{2\pi\sqrt{-1}\langle k,\sqrt{2v}B_t\rangle}\int_{[0,1)^2+\sqrt{2v}B_t} e^{-2\pi\sqrt{-1}\langle k,y\rangle}\omega(T-t,y)dy \\ &= e^{2\pi\sqrt{-1}\langle k,\sqrt{2v}B_t\rangle}\int_{[0,1)^2} e^{-2\pi\sqrt{-1}\langle k,y\rangle}\omega(T-t,y)dy \\ &= e^{2\pi\sqrt{-1}\langle k,\sqrt{2v}B_t\rangle}\omega\widehat{(T-t},\cdot)(k) \end{aligned}$$

here the third equality follows from the fact that ω is periodic. Substituting the above equality into (2.15) to obtain

$$X^1(t,x) = \frac{\sqrt{-1}}{2}\sum_{k=(k_1,k_2)\in\mathbb{Z}^2, k\neq 0}\frac{k_2}{|k|^2}e^{2\pi\sqrt{-1}\langle k,x\rangle}\widehat{Y(t,\cdot)}(k), \tag{2.16}$$

and

$$X^2(t,x) = -\frac{\sqrt{-1}}{2}\sum_{k\neq(k_1,k_2)\in\mathbb{Z}^2, k\neq 0}\frac{k_1}{|k|^2}e^{2\pi\sqrt{-1}\langle k,x\rangle}\widehat{Y(t,\cdot)}(k). \tag{2.17}$$

By our definition of linear operators K_i, the previous equations (2.16, 2.17) may be written as

$$X^j(t,x) = K_j(Y(t,\cdot))(x) \qquad \forall x \in \mathbb{R}^2,\ j=1,2. \tag{2.18}$$

Thanks to (2.18) we may finish our computation for Y as following. According to (2.11)

$$\xi(x) - Y(t,x) = \sqrt{2v}\int_t^T \langle Z(s,x), dB_s\rangle + \int_t^T \langle Z(s,x), K(Y(s,\cdot))(x)\rangle ds \tag{2.19}$$

where x runs through $\mathbb{R}^2$.

3 Feynman-Kac formula for the vorticity

The preceding stochastic integral equation (2.19) may be put in a differential form

$$dY = \langle Z, K(Y)\rangle dt + \sqrt{2v}\langle Z, dB\rangle, \quad Y_T = \xi \tag{3.1}$$

where the time space variable x, for simplicity, is suppressed. The initial value problem to the vorticity equation (2.3) is transferred to a terminal value problem to the stochastic differential equation (3.1) within the formulation of BSDEs. In order to derive a non-linear version of the Feynman-Kac formula for the vorticity ω, we need to study the infinite dimensional BSDE (3.1).

BSDE (3.1) possesses two features which make it difficulty to study. First, the stochastic equation (3.1) must be solved in a function space, so it is an infinite dimensional stochastic differential equation (with finite dimensional noise). Second, the non-linear term in this BSDE is quadratic in Y and Z, which is the origin of all difficulties. There are few results in literature about this kind of backward stochastic differential equations.

Let $B = (B^1, B^2)$ be a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_t^0 = \sigma\{B_s\colon\ s \le t\}$ and $(\mathcal{F}_t)_{t\ge 0}$ be the completed continuous filtration associated with $(\mathcal{F}_t^0)_{t\ge 0}$. Let $\mathcal{O}$ and $\mathcal{P}$ be the optional and predictable σ-fields on $\Omega \times [0,\infty)$, respectively. Let $\tilde{\mathcal{Q}} = \mathcal{O} \times \mathcal{B}(\mathbb{R}^2)$ and $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{B}(\mathbb{R}^2)$ be the optional and predictable σ-algebras on $\Omega\times[0,\infty)\times\mathbb{R}^2$. A $\tilde{\mathcal{Q}}$-measurable (resp. $\tilde{\mathcal{P}}$-measurable) function on $\Omega\times[0,\infty)\times\mathbb{R}^2$ is called a optional (resp. predictable) function, or called an $\mathbb{R}^2$-valued optional (resp. predictable). We may similarly define $\mathcal{O} \times \mathcal{B}(\mathbb{T}^2)$ and $\mathcal{P} \times \mathcal{B}(\mathbb{T}^2)$ which are identified with elements in the $\mathcal{O} \times \mathcal{B}(\mathbb{R}^2)$ and $\mathcal{P} \times \mathcal{B}(\mathbb{R}^2)$ respectively which are periodic in the space variables with period one.

In order to derive a non-linear version of the Feynman-Kac formula for the vorticity ω we need to prove the existence and the uniqueness of solutions to (3.1) subject to the given terminal value ξ. Actually we will do this for a general terminal value ξ which is not necessary in a form of $\varphi(B_T)$.

We will assume that ξ is a *bounded random function* on $\Omega \times \mathbb{T}^2$ which is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{T}^2)$ measurable, and furthermore, we assume that for every $w \in \Omega$, $\xi(w,\cdot) \in W^{2,2}(\mathbb{T}^2)$, and $\int_{\mathbb{T}^2} \xi(w,y)dy = 0$ for all $w \in \Omega$. In particular, according to the Sobolev imbedding, $x \to \xi(w,x)$ is continuous on $\mathbb{T}^2$ for every $w \in \Omega$.

By a solution to BSDE (3.1) we mean a pair of $\tilde{\mathcal{P}}$-measurable stochastic processes Y and Z, such that:

1) For all $x \in \mathbb{T}^2$, $(w,t) \to Y(w,t,x)$ is continuous semimartingale, and for all $(w,t) \in \Omega \times [0,T]$, $Y(w,t,\cdot) \in L^2(\mathbb{T}^2)$.

2) For every $x \in \mathbb{T}^2$,

$$\mathbb{E}\int_0^T |Z(t,x)|^2 dt < +\infty$$

so that the Itô's integral $\int_0^{\cdot} \langle Z(\cdot,x), dB\rangle$ is a square integrable martingale for every x.

3) It holds that

$$Y(t,x)=\xi(x)-\int_t^T \langle Z(s,x),K(Y(s,\cdot))(x)\rangle ds-\sqrt{2v}\int_t^T \langle Z(s,x),dB_s\rangle$$

almost surely on $\Omega\times\mathbb{T}^2$, for $t\in[0,T]$.

Now we are in a position to state our main result.

Theorem 3.1 *Under above assumptions on the terminal value ξ, there is a unique solution pair (Y,Z) to BSDE (3.1) such that*

1) Y is bounded on $\Omega\times[0,T]\times\mathbb{T}^2$, and

2) For almost all $x\in\mathbb{T}^2$, the Itô integral $\int_0^\cdot\langle Z(\cdot,x),dB\rangle$ is a BMO martingale, and

$$ess\sup_{[0,T]\times\Omega}\mathbb{E}\left\{\int_t^T ||Z_s||^2ds\middle|\mathcal{F}_t\right\}<+\infty$$

where $||\cdot||$ denotes the L^2-norm on $\mathbb{T}^2$.

In particular, by applying Theorem 3.1 to $\xi=\varphi(B_T)$ where $\varphi=\nabla\times u_0$ is bounded, C^2 on $\mathbb{T}^2$, then $Y(t,x)=\omega(T-t,\sqrt{2\nu}B_t+x)$, where (Y,Z) is the unique solution pair of (3.1) with terminal $\xi=\varphi(B_T)$ and ω is the solution to the vorticity equation (2.3) subject to the initial value $u(0,\cdot)=u_0$. Y may be regarded as the probabilistic representation for the vorticity ω.

The proof of Theorem 3.1 relies on two important technical facts. The first is the L^2-estimate for the linear operator K, and the secomd is a maximal principle formulated in terms of backward stochastic differential equations.

4 Several technical estimates

In order to prove the main result Theorem 3.1, we need several a priori estimates.

4.1 *Apriori* estimates for K

Let us recall the definition of K. Note that we identify tensor fields in the torus $\mathbb{T}^2$ with the tensor fields on $\mathbb{R}^2$ with period one along each space variable. For $k\in\mathbb{Z}_+$ and $q\geq 1$ the Sobolev space

$$\begin{aligned} W^{k,q}(\mathbb{T}^2) &= \{f:\partial^\alpha f\in L^q_{\mathrm{loc}}(\mathbb{R}^2)\cap L^q([0,1)^2) \text{ for } |\alpha|\leq k \\ &\quad \text{and } f(\cdot+e_i)=f(\cdot) \ \text{ for } i=1,2\} \end{aligned}$$

together with the Sobolev norm

$$||f||_{k,q}=\left(\sum_{\alpha\in\mathbb{Z}^2,|\alpha|\leq k}||\partial^\alpha f||_q^q\right)^{1/q}$$

where $||\cdot||_q$ is the L^q-norm over $\mathbb{T}^2$, that is

$$||f||_q = \left(\int_{\mathbb{T}^2} |f|^q\right)^{1/q} = \left(\int_{[0,1)^2} |f(x)|^q dx\right)^{1/q}.$$

If $q = 2$ then we use $||\cdot||$ instead of $||\cdot||_2$ for simplicity.

According to Sobolev's embedding theorem, $W^{2,2}(\mathbb{T}^2) \hookrightarrow C^\alpha(\mathbb{T}^2)$ for some $\alpha \in (0,1)$, so any element in $W^{2,2}(\mathbb{T}^2)$ has a unique continuous representation.

If $f \in L^2(\mathbb{T}^2)$ such that $\int_{[0,1)^2} f = 0$, then $K_j(f) = g_j$ are the unique solutions (with period one) such that $\int_{[0,1)^2} g_j = 0$, solving the Poisson equations

$$\Delta g_1 = -\frac{\partial f}{\partial x_2}, \;\; \Delta g_2 = \frac{\partial f}{\partial x_1} \quad \text{on } \mathbb{T}^2. \tag{4.1}$$

By definition, if $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, then $\partial^\alpha K_j(f) = K_j(\partial^\alpha f)$, where ∂^α stands for the partial derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$ for simplicity as long as $\partial^\alpha f \in L^2(\mathbb{T}^2)$.

On the other hand

$$\int_{\mathbb{T}^2} |\nabla g_j|^2 = -\int_{\mathbb{T}^2} g_j \Delta g_j = \int_{\mathbb{T}^2} g_1 \frac{\partial f}{\partial x_2} \text{ or } -\int_{\mathbb{T}^2} g_2 \frac{\partial f}{\partial x_1}$$

according to $j = 1$ or $j = 2$. Integrating by parts together with Cauchy-Schwartz inequality to the last integrals we deduce that

$$\int_{\mathbb{T}^2} |\nabla g_j|^2 \le \sqrt{\int_{\mathbb{T}^2} |\nabla g_j|^2} \sqrt{\int_{\mathbb{T}^2} |f|^2}$$

which yields that

$$||\nabla K_j(f)|| \le ||f||, \qquad j = 1, 2. \tag{4.2}$$

Let $\lambda_1 > 0$ be the spectral gap for the torus $\mathbb{T}^2$. Since $\int_{\mathbb{T}^2} K_j(f) = 0$, according to the Poincaré inequality

$$||K_j(f)|| \le \frac{1}{\sqrt{\lambda_1}} ||\nabla K_j(f)|| \le \frac{1}{\sqrt{\lambda_1}} ||f||. \tag{4.3}$$

Therefore we have the following elliptic estimate (for more details see for example [1], [2] and [10]).

Lemma 4.1 *There is a universal constant $C_0 > 0$ such that*

$$||K_j(f)||_{k,2} \le C_0 ||f||_{k-1,2}$$

for any $f \in W^{k-1,2}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} f = 0$, where $k \in \mathbb{N}$.

In particular, if $f \in W^{1,2}(\mathbb{T}^2)$, $K(f)$ is α-Hölder continuous.

4.2 A maximum principle

Let us formulate a probabilistic version of the maximum principle in terms of BSDE.

Lemma 4.2 *Suppose y is a continuous semimartingale such that*

$$y_t = y_T - \int_t^T \langle h, z\rangle ds - \int_t^T \langle z, dB\rangle \quad \textit{for } t \in [0,T],$$

where y_T is a bounded $\mathcal{F}_T$-measurable random variable, both z and h are $\mathbb{R}^d$-valued predictable processes such that

$$\mathbb{E}\int_0^T |z|^2 dt < \infty$$

and suppose that

$$R_t = \exp\left[-\int_0^t \langle h, dB\rangle - \frac{1}{2}\int_0^t |h|^2 ds\right]$$

is a martingale up to T. Then $|y_t|_\infty \le |y_T|_\infty$ for all $t \in [0,T]$ almost surely.

Proof. Define a probability $\mathbb{Q}$ with density R. Then $\mathbb{P}$ is equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, and according to Girsanov's theorem $\tilde{B}_t = B_t + \int_0^t h_s ds$ is a standard Brownian motion, and

$$y_t - y_T = -\int_t^T \langle z, d\tilde{B}\rangle.$$

Therefore $y_t = \mathbb{E}^{\mathbb{Q}}\left\{y_T | \mathcal{F}_t\right\}$ so that $|y_t|_\infty \le |y_T|_\infty$ almost surely. ■

4.3 A linear BSDE

Let us consider the following linear BSDE

$$dY = \langle Z, h\rangle dt + \langle Z, dB\rangle, \;\; Y_T = \xi \tag{4.4}$$

with $h \in \tilde{\mathcal{Q}}$ is a given optional process (valued in $\mathbb{T}^2$) such that for each $(w,t) \in \Omega \times [0,T]$, $h(w,t,\cdot) \in C(\mathbb{T}^2)$ and

$$\mathbb{E}\int_0^T |h(t,x)|^2 dt < \infty \quad \forall x \in \mathbb{T}^2. \tag{4.5}$$

ξ is the terminal value:

$$\xi \in L^\infty(\Omega \times \mathbb{T}^2) \cap L^\infty(\Omega, \mathcal{F}_T, W^{2,2}(\mathbb{T}^2)).$$

The linear equation (4.4) may be solved for every $x \in \mathbb{T}^2$, and in fact we may solve the linear BSDE

$$\begin{aligned} dY(t,x) &= \langle Z(t,x), h(t,x)\rangle dt + \langle Z(t,x), dB_t\rangle, \\ Y(T,x) &= \xi(x), \end{aligned} \tag{4.6}$$

by means of changing probability. More precisely, for each $x \in \mathbb{T}^2$, since (4.5) holds, we can define a probability $\mathbb{Q}^x$ on $\mathcal{F}_T$ by $\frac{d\mathbb{Q}^x}{d\mathbb{P}} = R(T,x)$, where

$$R(t,x) = \exp\left[-\int_0^t \langle h(s,x), dB_s\rangle - \frac{1}{2}\int_0^t |h(s,x)|^2 ds\right].$$

If $(Y(\cdot,x), Z(\cdot,x))$ is the unique solution of (4.6), then, according to the Girsanov theorem, $Y(\cdot,x)$ must be a martingale under the new probability $\mathbb{Q}^x$, we therefore have

$$Y(t,x) = \mathbb{E}^{\mathbb{Q}^x}\left\{\xi(x)|\mathcal{F}_t\right\}$$

which implies that

$$Y(t,x) = \mathbb{E}\left\{\frac{R(T,x)}{R(t,x)}\xi(x)\bigg|\mathcal{F}_t\right\}$$

for $(t,x) \in [0,T] \times \mathbb{T}^2$. Therefore we have established the following

Lemma 4.3 *Suppose that ξ is $W^{2,2}(\mathbb{T}^2)$-valued $\mathcal{F}_T$-measurable random variable, and suppose that h is a $W^{2,2}(\mathbb{T}^2)$-valued adapted stochastic process satisfying (4.5), then the unique solution to (4.4) is given by*

$$Y(t,x) = \mathbb{E}\left\{\xi(x)e^{-\int_t^T\langle h(s,x),dB_s\rangle - \frac{1}{2}\int_t^T |h(s,x)|^2 ds}\bigg|\mathcal{F}_t\right\} \tag{4.7}$$

for $t \in [0,T] \times \mathbb{T}^2$.

It is clear that

$$\begin{aligned}\partial_j Y(t,x) &= \mathbb{E}\left\{\partial_j\xi(x)\frac{R(T,x)}{R(t,x)}\bigg|\mathcal{F}_t\right\}\\ &\quad+\mathbb{E}\left\{\xi(x)\left(-\int_t^T\langle\partial_j h(s,x), dB_s\rangle - \int_t^T\langle h(s,x), \partial_j h(s,x)\rangle ds\right)\frac{R(T,x)}{R(t,x)}\bigg|\mathcal{F}_t\right\}\end{aligned}$$

so we have the following simple fact.

Corollary 4.4 *1) If in addition ξ and h are bounded, then the solution Y is continuous in (t,x). 2) If in addition ξ and h have bounded derivatives in x, then so is Y.*

5 Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1. We will use the following convention in our proof. The elliptic estimates show that if $f \in W^{k,2}(\mathbb{T}^2)$ then $K(f) \in W^{k+1,2}(\mathbb{T}^2)$, thus if $k \geq 1$, then, according to the Sobolev imbedding, $K(f)$ has a Hölder continuous version. Therefore, if $f \in W^{1,2}(\mathbb{T}^2)$ for $k \geq 1$, $K(f)$ is always chosen to be its continuous version.

Let $\mathcal{H}$ denote the collection of all bounded $\tilde{\mathcal{P}}$-measurable stochastic processes Y on $\Omega \times [0,T] \times \mathbb{T}^2$ satisfying the following conditions:

1) For each $x \in \mathbb{T}^2$, $Y(\cdot, x)$ is a continuous semimartingale (up to time T) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ whose martingale part M (with $M_0 = 0$) is a BMO martingale, and $Y_T = \xi$. Moreover, for every $(w,t) \in \Omega \times [0,T]$, $Y(w,t,\cdot) \in W^{2,2}(\mathbb{T}^2)$.

2) Let the Itô representation of the martingale part

$$M(t,x) = \int_0^t \langle Z(t,x), dB_t \rangle$$

where Z is $\tilde{\mathcal{P}}$-measurable. Then

$$\operatorname*{ess\ sup}_{[0,T]\times\Omega} \mathbb{E}\left\{ \int_t^T ||Z_s||^2 ds \middle| \mathcal{F}_t \right\} < +\infty.$$

Let $Y \in \mathcal{H}$, we define $\mathcal{L}(Y) = \tilde{Y}$ by solving the following *linear* backward stochastic differential equation

$$d\tilde{Y}(t,x) = \langle \tilde{Z}(t,x), K(Y(t,\cdot))(x) \rangle dt + \sqrt{2v} \langle \tilde{Z}(t,x), dB_t \rangle, \ \tilde{Y}(T,x) = \xi(x). \tag{5.1}$$

for every $x \in \mathbb{T}^2$. Then $\tilde{Y} \in \mathcal{H}$.

5.1 *Apriori* estimate for the density process Z

If $Y \in \mathcal{H}$, in particular Y is a bounded function on $\Omega \times [0,T] \times \mathbb{T}^2$. $||Y||_\infty$ denotes the essential bound of Y on $\Omega \times [0,T] \times \mathbb{T}^2$.

Suppose $Y \in \mathcal{H}$ such that $||Y||_\infty \le C_1$. Define $\tilde{Y} = \mathcal{L}(Y)$, and $\tilde{Z}$ is the density process of the martingale part of $\tilde{Y}$, that is, $(\tilde{Y}, \tilde{Z})$ by solving the following linear BSDE

$$d\tilde{Y} = \langle \tilde{Z}, K(Y) \rangle ds + \sqrt{2\nu} \langle \tilde{Z}, dB \rangle, \ \tilde{Y}_T = \xi \tag{5.2}$$

where $|\xi(w,t,x)| \le C_1$. By the maximal principle, $|\tilde{Y}(w,t,x)| \le C_1$.

For simplicity, we will use $\mathbb{E}^{\mathcal{F}_t}$ to denote the conditional expectation $\mathbb{E}\{\cdot|\mathcal{F}_t\}$.

By Itô's calculus,

$$\begin{aligned} |\tilde{Y}_t|^2 &= |\xi|^2 - 2\nu \int_t^T |\tilde{Z}|^2 ds - 2 \int_t^T \tilde{Y} \langle \tilde{Z}, K(Y) \rangle ds \\ &\quad -2\sqrt{2\nu} \int_t^T \tilde{Y} \langle \tilde{Z}, dB \rangle \ . \end{aligned}$$

First take conditional expectations, to obtain that

$$
\begin{aligned}
|\tilde{Y}_t|^2 + 2\nu\mathbb{E}^{\mathcal{F}_t}\int_t^T |\tilde{Z}|^2 ds &= \mathbb{E}^{\mathcal{F}_t}|\xi|^2 - 2\mathbb{E}^{\mathcal{F}_t}\int_t^T \tilde{Y}\langle \tilde{Z}, K(Y)\rangle ds \\
&\leq C_1^2 + 2C_1\mathbb{E}^{\mathcal{F}_t}\int_t^T |\langle \tilde{Z}, K(Y)\rangle| ds
\end{aligned}
$$

then integrating over $\mathbb{T}^2$ and using the estimate from the maximum principle, we have

$$
\begin{aligned}
& ||\tilde{Y}_t||^2 + 2\nu\mathbb{E}^{\mathcal{F}_t}\int_t^T ||\tilde{Z}||^2 ds \\
\leq\ & C_1^2 + 2C_1\mathbb{E}^{\mathcal{F}_t}\int_t^T \int_{\mathbb{T}^2} |\langle \tilde{Z}, K(Y)\rangle| ds \\
\leq\ & C_1^2 + 2C_1\mathbb{E}^{\mathcal{F}_t}\int_t^T ||K(Y)|| ||\tilde{Z}|| ds \\
\leq\ & C_1^2 + C_1\mathbb{E}^{\mathcal{F}_t}\int_t^T \left[\varepsilon ||K(Y)||^2 + \frac{1}{\varepsilon}||\tilde{Z}||^2\right] ds \\
\leq\ & C_1^2 + \varepsilon C_1 C_0 \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T ||Y||^2 ds\right] + \frac{C_1}{\varepsilon}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T ||\tilde{Z}||^2 ds\right].
\end{aligned}
$$

for every $\varepsilon > 0$. Recall that

$$
||\tilde{Z}||_{BMO}^2 = \text{ess} \sup_{\Omega\times[0,T]} \mathbb{E}^{\mathcal{F}_t}\int_t^T ||\tilde{Z}||^2 ds \ .
$$

It follows that

$$
||\tilde{Z}||_{BMO}^2 \leq \frac{C_1}{2\nu}\left[C_1 + T\varepsilon C_1^2 C_0 + \frac{1}{\varepsilon}||\tilde{Z}||_{BMO}^2\right]. \tag{5.3}
$$

Choose $\varepsilon = \frac{C_1}{\nu}$, we obtain

$$
||\tilde{Z}||_{BMO} \leq \frac{C_1}{\nu}\sqrt{\nu + TC_0C_1^2} \ .
$$

That is, the norms $||\tilde{Y}||_\infty$ and $||\tilde{Z}||_{BMO}$ are uniformly bounded, depending only on ν, C_1, C_0 and T.

5.2 Contraction property

Let α be a real number to be chosen later, and consider $Y_t^\alpha = e^{\alpha t}Y_t$ and $\tilde{Y}_t^\alpha = e^{\alpha t}\tilde{Y}_t$. Then, according to integration by parts

$$
d\tilde{Y}^\alpha = \langle \tilde{Z}, K(Y^\alpha)\rangle ds + \sqrt{2\nu}\langle \tilde{Z}^\alpha, dB\rangle + \alpha\tilde{Y}^\alpha dt.
$$

Denoting $\delta Y^\alpha = Y^\alpha - Y'^\alpha$ and $\delta Z^\alpha = Z^\alpha - Z'^\alpha$. Then

$$d(\delta\tilde{Y}^\alpha) = \Phi ds + \alpha(\delta\tilde{Y}^\alpha)ds + \sqrt{2\nu}\langle\delta\tilde{Z}^\alpha, dB\rangle$$

where

$$\Phi_s = \langle\tilde{Z}_s, K(Y_s^\alpha)\rangle - \langle\tilde{Z}'_s, K(Y_s'^\alpha)\rangle\ .$$

It follows that

$$\begin{aligned}|\delta\tilde{Y}_t^\alpha|^2 &= -2\nu\int_t^T|\delta\tilde{Z}^\alpha|^2ds - 2\alpha\int_t^T|\delta\tilde{Y}^\alpha|^2ds \\ &\quad -2\int_t^T(\delta\tilde{Y}^\alpha)\Phi ds - 2\sqrt{2\nu}\int_t^T(\delta\tilde{Y}^\alpha)\langle\delta\tilde{Z}^\alpha, dB\rangle\end{aligned}$$

by taking conditional expectation given the information up to $\mathcal{F}_t$ to obtain

$$\begin{aligned}|\delta\tilde{Y}_t^\alpha|^2 &= -2\nu\mathbb{E}^{\mathcal{F}_t}\int_t^T|\delta\tilde{Z}^\alpha|^2ds - 2\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T|\delta\tilde{Y}^\alpha|^2ds \\ &\quad -2\mathbb{E}^{\mathcal{F}_t}\int_t^T(\delta\tilde{Y}^\alpha)\Phi ds.\end{aligned}$$

Now integrating over $\mathbb{T}^2$, to obtain

$$\begin{aligned}||\delta\tilde{Y}_t^\alpha||^2 &= -2\nu\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Z}^\alpha||^2ds - 2\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||^2ds \\ &\quad -2\mathbb{E}^{\mathcal{F}_t}\int_t^T\int_{\mathbb{T}^2}(\delta\tilde{Y}^\alpha)\Phi ds\end{aligned}\tag{5.4}$$

Let us write for simplicity

$$J(t) = ||\delta\tilde{Y}_t^\alpha||^2 + 2\nu\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Z}^\alpha||^2ds + 2\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||^2ds.$$

Then (5.4) implies that

$$\begin{aligned}J(t) &= -2\mathbb{E}^{\mathcal{F}_t}\int_t^T\int_{\mathbb{T}^2}(\delta\tilde{Y}^\alpha)\Phi ds \\ &\leq 2\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||||\Phi||ds \\ &\leq 2\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||^2ds\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T||\Phi||^2ds\right)^{\frac{1}{2}}\end{aligned}\tag{5.5}$$

which yields that

$$\begin{aligned}J(t) &\leq 2\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||^2ds\right)^{\frac{1}{2}}\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T||\Phi||^2ds\right)^{\frac{1}{2}} \\ &\leq 2\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T||\delta\tilde{Y}^\alpha||^2ds + \frac{1}{2\alpha}\mathbb{E}^{\mathcal{F}_t}\int_t^T||\Phi||^2ds.\end{aligned}\tag{5.6}$$

Let us now consider the last integral appearing on the right-hand side of (5.6). It is clear that

$$\begin{aligned}
||\Phi_s|| &= ||\tilde{Z}_s \cdot K(Y_s^\alpha) - \tilde{Z}'_s \cdot K(Y_s'^\alpha)|| \\
&= ||\tilde{Z}_s \cdot K(\delta Y_s^\alpha) + \delta\tilde{Z}_s^\alpha \cdot K(Y'_s)|| \\
&\le ||K(\delta Y_s^\alpha)||||\tilde{Z}_s|| + ||K(Y'_s)||||\delta\tilde{Z}_s^\alpha|| \\
&\le C_0||\delta Y_s^\alpha||||\tilde{Z}_s|| + C_0||Y'_s||||\delta\tilde{Z}_s^\alpha||
\end{aligned}$$

plugging into (5.6) we conclude that

$$\begin{aligned}
J(t) &\le 2\alpha\mathbb{E}^{\mathcal{F}_t}\int_t^T ||\delta\tilde{Y}^\alpha||^2 ds + \frac{C_0^2C_1^2}{\alpha}\frac{\nu + TC_0C_1^2}{\nu^2}||\delta Y^\alpha||_\infty^2 \\
&\quad + \frac{C_0^2C_1^2}{\alpha}\mathbb{E}^{\mathcal{F}_t}\int_t^T ||\delta\tilde{Z}^\alpha||^2 ds
\end{aligned} \tag{5.7}$$

where we have used the uniform bounds

$$||\tilde{Y}||_\infty \le C_1 \text{ and } ||\tilde{Z}||_{BMO} \le \frac{C_1}{\nu}\sqrt{\nu + TC_0C_1^2}\ .$$

Choose $\alpha > 0$ such that

$$\frac{C_0^2C_1^2}{\alpha}\frac{\nu + TC_0C_1^2}{\nu^2} \le \frac{1}{16},\ \frac{C_0^2C_1^2}{\alpha} \le \frac{\nu}{4}$$

then (5.7) yields that

$$||\delta\tilde{Y}^\alpha||_\infty + ||\delta\tilde{Z}^\alpha||_{BMO} \le \frac{1}{2}||\delta Y^\alpha||_\infty\ .$$

Theorem 5.1 *There is $\alpha > 0$ such that, $\mathcal{L}$ is a contraction on $\mathcal{H}$ under the norm*

$$||Y||_{\alpha,\infty} = ||Y^\alpha||_\infty + ||Z^\alpha||_{BMO}$$

where $Z_t^\alpha = e^{\alpha t}Z_t$ and Z is the density process of the martingale part of Y.

We are now in a position to complete the proof of Theorem 3.1. The sequence of Picard's iteration is constructed as the following. Begin with

$$Y_0(t,x) = \mathbb{E}\left\{\xi(x)|\,\mathcal{F}_t\right\}$$

(here we mean the continuous version of the optional projection of ξ) and Z_0 is the density process of Y_0 with respect to the Brownian motion determined by Itô's martingale representation. Since $\xi \in W^{2,2}(\mathbb{T}^2)$, so $Y_0(t,\cdot) \in W^{2,2}(\mathbb{T}^2)$ for all t almost surely. Define $Y_{n+1} = \mathcal{L}(Y_n)$ for $n = 0, 1, 2, \cdots$. Then Lemma 4.3 implies that all $Y_n \in \mathcal{H}$ and in particular $(t,x) \to Y_n(\cdot,t,x)$ are continuous almost surely, so that

$$\mathbb{P}\{|Y_n(t,x)| \le C_1 \text{ for all } (t,x,n) \in [0,T]\times\mathbb{T}^2\times\mathbb{N}\} = 1. \tag{5.8}$$

Theorem 5.1 implies that $\{Y_n\}$ is a Cauchy sequence under the norm $||\cdot||_{\alpha,\infty}$ for some $\alpha > 0$, and therefore has a limit Y which is a solution to (3.1).

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727. MR 0125307 (23 #A2610)

[2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92. MR 0162050 (28 #5252)

[3] J.-M. Bismut, *Théorie probabiliste du contrôle des diffusions*, Mem. Amer. Math. Soc. **4** (1976), no. 167, xiii+130. MR 0453161 (56 #11428)

[4] P. Briand and Y. Hu, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, Probab. Theory Related Fields **141** (2008), no. 3-4, 543–567. MR 2391164 (2009e:60133)

[5] A. B. Cruzeiro and E. Shamarova, *Navier-Stokes equations and forward-backward SDEs on the group of diffeomorphisms of the torus*, Stochastic Processes and their Applications 119 (2009), 4034-4060.

[6] F. Delarue, *Estimates of the solutions of a system of quasi-linear PDEs. A probabilistic scheme*, Séminaire de Probabilités XXXVII, Lecture Notes in Math., vol. 1832, Springer, Berlin, 2003, pp. 290–332. MR 2053051 (2005b:60165)

[7] R. P. Feynman, *The Principle of Least Action in Quantum Mechanics*, in Feynman's Thesis: A New Approach to Quantum Theory, Ed. by L. M. Brown, World Scientific (2005).

[8] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab. **28** (2000), no. 2, 558–602. MR 1782267 (2001h:60110)

[9] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 1, 193–248. MR 1555394

[10] C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag New York, Inc., New York, 1966. MR 0202511 (34 #2380)

[11] É. Pardoux and S. G. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett. **14** (1990), no. 1, 55–61. MR 1037747 (91e:60171)

[12] S. G. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stochastics Rep. **37** (1991), no. 1-2, 61–74. MR 1149116 (93a:35159)

[13] R. Tevzadze, *Solvability of backward stochastic differential equations with quadratic growth*, Stochastic Process. Appl. **118** (2008), no. 3, 503–515. MR 2389055 (2009d:60194)